ISRAEL JOURNAL OF MATHEMATICS 99 (1997), 109–121

THE DISCONNECTION EXPONENT FOR SIMPLE RANDOM WALK

BY

GREGORY F. LAWLER*

Department of Mathematics, Box 90320
Duke University, Durham, NC 27708-0320, USA
e-mail: jose@math.duke.edu

AND

EMILY E. PUCKETTE

Department of Mathematics, Occidental College
Los Angeles, CA 90041, USA
e-mail: eep@oxy.edu

ABSTRACT

Let $S(t)$ denote a simple random walk in Z^2 with integer time t. The disconnection exponent $\bar{\gamma}$ is defined by saying the probability that the path of S starting at 0 and ending at the circle of radius n disconnects 0 from infinity decays like $n^{-\bar{\gamma}}$. We prove that the disconnection exponent is well-defined and equals the disconnection exponent for Brownian motion which is known to exist.

1. Introduction

Let $S(t)$ denote a simple nearest neighbor random walk in Z^2 with integer time t. The disconnection exponent $\bar{\gamma}$ is defined by saying that the probability that the path of a random walker from 0 to the sphere of radius n does not disconnect

* Research supported by the National Science Foundation.
Received November 8, 1994 and in revised form September 12, 1995

0 from the sphere is of order $n^{-\bar{\gamma}}$. To be more precise let C_n be the discrete ball of radius n,

$$C_n = \{x \in Z^2 \colon |x| < n\},$$

and

$$\partial C_n = \{y \in Z^2 \setminus C_n \colon |y - x| = 1 \text{ for some } x \in C_n\}.$$

Let

$$\tau_n = \inf\{t \geq 0 \colon S(t) \in \partial C_n\}$$

and let $\bar{S}_n$ be the event that there is a nearest neighbor path from the origin to ∂C_n which does not intersect

$$S(0, \tau_n] = \{S(t) \colon 0 < t \leq \tau_n\}.$$

We then define $\bar{\gamma}$ by

$$P(\bar{S}_n) \approx n^{-\bar{\gamma}},$$

where $\approx$ denotes "logarithmically asymptotic to", i.e.,

$$\lim_{n \to \infty} \frac{\ln P(\bar{S}_n)}{\ln n} = -\bar{\gamma}.$$

Implicit in this definition is the existence of the limit. In [4] it was shown that $\bar{\gamma} > 0$ in the sense that

$$\liminf_{n \to \infty} -\frac{\ln P(\bar{S}_n)}{\ln n} > 0,$$

but it was not shown that the limit exists. The purpose of this paper is to show that the limit exists and is equal to the corresponding exponent for Brownian motion.

Let $B(t)$ denote a Brownian motion in R^2. Let D_r denote the open ball of radius r about 0 with boundary ∂D_r. Set

$$T_r = \inf\{t \geq 0 \colon B(t) \in \partial D_r\}.$$

Start a Brownian motion on the ball of radius 1 about the origin. For $r > 1$, let $\phi(r)$ be the probability that the origin is in the unbounded component of the complement of $B[0, T_r]$, i.e., the probability that the Brownian motion does not disconnect the origin from ∂D_r. A straightforward subadditivity argument (see [9, 5.5]) can be used to show that the limit

$$\lim_{r \to \infty} -\frac{\ln \phi(r)}{\ln r} = \gamma$$

exists, i.e., $\phi(r) \approx r^{-\gamma}$. The exact value of γ is not known although it has been conjectured using nonrigorous conformal field theory [6] that $\gamma = 1/4$. It has been shown that

$$\frac{1}{2\pi} \leq \gamma < .469$$

The upper bound for γ has been proven by Werner [11,12]. The lower bound was derived by Burdzy and Lawler [2, 3] and used to estimate the "intersection exponent" for Brownian motion, which they showed is the same as the intersection exponent for simple random walk. Cranston and Mountford [5] have shown that this is also the intersection exponent for any mean zero, finite variance, truly 2-dimensional random walk.

In trying to estimate intersection probabilities for a model of random walks in dimensions strictly between one and two, the second author [10] wanted to use the disconnection exponent to help estimate some intersection exponents. This requires working with $\bar{\gamma}$ rather than γ. The purpose of this paper is to show that they are in fact the same.

THEOREM 1.1: *Let $S(n)$ be a simple random walk in Z^2 and let $\bar{S}_n$ be the event described above. Then*

$$P(\bar{S}_n) \approx n^{-\gamma},$$

where γ denotes the disconnection exponent for Brownian motion.

The proof uses a strong approximation theorem which states that a Brownian motion and a random walk can be defined on the same probability space so that up to distance n the paths differ by at most $c \ln n$ except for an event of very small probability. The one-dimensional version of this result was proved by Komlós, Major, and Tusnády [8] and a version for simple random walk in two dimensions was discussed by Auer [1].

In the next section we outline the facts about random walks and Brownian motion that we will need. The proof of the main theorem will appear in the final section.

ACKNOWLEDGEMENT: The authors would like to thank the referee for useful comments that simplified some of the arguments in this paper.

2. Preliminary results

In this section we will state a number of results about random walks and Brownian motion that we will need in our proof. We start with a strong approximation theorem for random walk and Brownian motion first proved in [8]. The following version for simple random walk in Z^2 can be found in [1].

LEMMA 2.1: *For every $\lambda > 0$, there exists a constant $c = c(\lambda) < \infty$ such that a simple random walk S on Z^2 and a Brownian motion B with $B(0) = S(0)$ can be defined on the same probability space such that for each n,*

$$P\{\max_{0\le t\le n} \sup_{|t-s|\le \frac{1}{2}} |S(t) - B(s)| \ge c\ln n\} = O(n^{-\lambda}).$$

This lemma can be modified into the following lemma which we will use. Recall the definitions of T_n and τ_n from the previous section. It is easy to check that

$$P\{T_n \ge n^3\} \le e^{-an}, \quad P\{\tau_n \ge n^3\} \le e^{-an},$$

for some $a > 0$. We therefore can conclude this version of the lemma.

LEMMA 2.2: *For any $\lambda > 0$, there exists a constant $c = c(\lambda)$ such that a simple random walk S on Z^2 and a Brownian motion B with $B(0) = S(0)$ can be defined on the same probability space such that for each n,*

$$P\{\max_{0\le t\le \bar{T}_n} \sup_{|t-s|\le \frac{1}{2}} |S(t) - B(s)| \le c\ln n\} \ge 1 - c'n^{-\lambda}$$

for some constant $c' > 0$ and for $\bar{T}_n = \max(T_n, \tau_n)$.

We now let $\kappa = 2c(100)$ in Lemma 2.2. (We will never need this sharp an estimate, but it will make the proof easier if we fix a particular κ.)

Also, we will need lower bounds on the random walk disconnection probability. Arguments similar to those in [4, Lemmas 5,6] prove the following lemma. Let P^x denote that the simple random walk starts at the point x.

LEMMA 2.3: *Let $\bar{S}_n$ be the event*

$$\{S[0, \tau_n] \text{ does not disconnect } 0 \text{ and } \partial C_n\}.$$

(a) *There exists a constant $c > 0$ such that for all $|x| < (n/2) + 1$,*

$$P^x(\bar{S}_n^c) > c.$$

(b) *There exist constants $\lambda > 0$ and $c < \infty$ such that for all $|x| < (n/2) + 1$,*

$$P^x(\bar{S}_n^c) \geq 1 - c\,(|x|/n)^\lambda.$$

3. Proof of Theorem

We will let $\bar{B}_n$ denote the event

$$\{B[0, T_n] \text{ does not disconnect } 0 \text{ and } \partial D_n\},$$

and for $r < n$,

$$\phi(r, n) = P^x(\bar{B}_n),$$

where $|x| = r$. Note that by scaling, $\phi(r, n) = \phi(n/r)$, for the ϕ defined in the first section. For $r < n$, let

$$\psi(r, n) = \sup_{x \in \partial C_r} P^x(\bar{S}_n).$$

Finally we let κ be the constant defined after Lemma 2.2. We let $W_n[s, t]$ be the closed Wiener sausage of radius $\kappa \ln n$ about $B[s, t]$,

$$W_n[s, t] = \{y \in R^2 \colon |y - B(u)| \leq \kappa \ln n \text{ for some } u \in [s, t]\}.$$

We then let $\bar{W}_n$ denote the event

$$\{W_n[0, T_n] \text{ does not disconnect } 0 \text{ and } \partial D_n\},$$

and for $r < n$,

$$\bar{\phi}(r, n) = P^x(\bar{W}_n),$$

for $|x| = r$. Note that this probability is independent of which x in the circle of radius r we choose; however, since the radius of the Wiener sausage depends on n, it is not true that

$$\bar{\phi}(r, n) = \bar{\phi}\left(1, \frac{n}{r}\right).$$

The forthcoming lemma tells us that in some sense the asymptotic behavior of $\bar{\phi}(r, n)$ is the same as that of $\phi(r, n)$. Recall that γ, defined in Section 1, is the disconnection exponent for Brownian motion, and hence for any $a \in (0, 1)$,

$$\phi(n^a, n) = \phi(n^{1-a}) \approx n^{-(1-a)\gamma}.$$

LEMMA 3.1: *For every $0 < a < 1$, there exist $\delta > 0$ and $c < \infty$ such that*

$$\phi(n^a, 2n) - c\, n^{-(1-a)\gamma-\delta} \le \bar{\phi}(n^a, n) \le \phi(n^a, n).$$

Proof: Fix $0 < a < 1$. Since $\bar{W}_n \subset \bar{B}_n$, we have, for $|x| = n^a$,

$$\begin{aligned} P^x(\bar{B}_n) &\ge P^x(\bar{W}_n) = P^x(\bar{B}_n \cap \bar{W}_n) \\ &\ge P^x(\bar{B}_{2n} \cap \bar{W}_n) = P^x(\bar{B}_{2n}) - P^x(\bar{B}_{2n} \cap \bar{W}_n^c). \end{aligned}$$

Then, as it is easy to show that there exist δ, c such that

$$\phi(2\kappa \ln n, n) \le c\, n^{-(1-a)\gamma-\delta},$$

it suffices to show that

$$P^x(\bar{B}_{2n} \cap \bar{W}_n^c) \le \phi(2\kappa \ln n, n).$$

First suppose that $T_{2\kappa \ln n} < T_n$. Then the strong Markov property implies that at time $T_{2\kappa \ln n}$,

$$P^x((\bar{B}_{2n} \cap \bar{W}_n^c) \cap \{T_{2\kappa \ln n} < T_n\}) \le P^x\{T_{2\kappa \ln n} < T_n\}\, \phi(2\kappa \ln n, n).$$

Conversely, now suppose that $T_{2\kappa \ln n} > T_n$. Let

$$\sigma = \sigma_n = \inf\{t \ge 0 \colon W_n[0,t] \text{ disconnects } 0 \text{ and } \partial D_n\}.$$

Then σ is a stopping time and in the event $\bar{W}_n^c$, we have $\sigma \le T_n$. Furthermore, if $T_{2\kappa \ln n} > T_n$ and $\bar{W}_n^c$ occurs, then there exists a time

$$\sigma' = \sup\{t \in [0, \sigma) \text{ such that } W_n[\sigma', \sigma] \text{ disconnects } 0 \text{ and } \partial D_n\}.$$

Then $|B(\sigma') - B(\sigma)| \le 2\kappa \ln n$ by definition of W_n, and it is easy to see that $B[\sigma', \sigma] \cup [B(\sigma'), B(\sigma)]$ disconnects 0 from ∂D_n. Also from topological considerations, one can see that if $B[\sigma, T_{2n}]$ disconnects $B(\sigma')$ from ∂D_{2n}, then $B[0, T_{2n}]$ disconnects 0 from ∂D_{2n}.

Consider the event E that $B[\sigma, T_{2n}]$ does not disconnect $B(\sigma')$ from ∂D_{2n} (conditional on $T_{2\kappa \ln n} > T_n$ and $\bar{W}_n^c$). Here we have a Brownian motion starting at distance $2\kappa \ln n$ from the point $B(\sigma')$ and reaching a distance of at least n without disconnecting $B(\sigma')$ from the infinity. Thus $P(E) \le \phi(2\kappa \ln n, n)$.

Then,

$$\begin{aligned} P^x(\bar{B}_{2n} \cap \bar{W}_n^c) &\le \phi(2\kappa \ln n, n)(P^x(T_{2\kappa \ln n} < T_n) + P^x(T_{2\kappa \ln n} > T_n)) \\ &= \phi(2\kappa \ln n, n) \end{aligned}$$

which completes the proof. ∎

We will next prove the upper bound

$$\limsup_{n\to\infty} -\frac{\ln P^0(\bar{S}_n)}{\ln n} \leq \gamma.$$

To show this, we will need lower bounds on $P^0(\bar{S}_n)$, so we will consider a specific case of no disconnection from the origin to the boundary of the ball of radius n. First we require that the random walk S stay within a sector of the ball C_{n^a} until it reaches ∂C_{n^a} which will ensure no disconnection from time 0 to time τ_{n^a}. Second, starting a Wiener sausage at the point $S(\tau_{n^a})$, we require that the random walk stay within this sausage until it hits the boundary ∂C_n and that the Wiener sausage does not disconnect 0 and ∂D_n. This ensures that there is no disconnection by the random walk from time τ_{n^a} to time τ_n. To keep the two parts of the random walk from creating a disconnection, we also require that the generating Brownian motion of the Wiener sausage stay outside the ball $D_{n^a+\kappa \ln n}$ after a certain time ρ.

Consider the discrete ball C_{n^a} in Z^2 and the half-plane

$$A = \{(x,y) \in R^2 \colon x \geq 0\}.$$

The set $C_{n^a} \cap A$ is a sector of the ball subtended by the angle $\theta = \pi$. If we consider a random walk S starting at the origin and define the event

$$U = \{S[0, \tau_{n^a}] \subset (C_{n^a} \cap A)\},$$

then (see, for example, [9, Proposition 2.4.5]) there exists a constant c such that

$$P^0(U) \geq cn^{-a}.$$

Note that if the random walk S stays in this sector, $S[0, \tau_{n^a}]$ cannot disconnect 0 and ∂D_n.

Next consider the ray $L = \{(x,y) \in R^2 \colon x \leq 0, y = 0\}$. Let B be a Brownian motion starting at some point x on the set $\partial D_{n^a} \cap A$ and define the stopping time

$$\rho = \inf\{t > 0 \colon B(t) \in L\}.$$

Consider the event

$$X = \{|B(t)| > n^a + \kappa \ln n,\ t \in [\rho, T_n]\}.$$

Let w be a point on ∂D_{2n^a}. And let $G = [\arg(B(0)) - \frac{\pi}{4}, \arg(B(0)) + \frac{\pi}{4}]$. Then there exists a constant c such that

$$\begin{aligned} P^x(X) &\geq P^x\{\arg(B(t)) \in G,\ t \in [0, T_{2n^a}]\}\ P^w\{T_n < T_{\frac{3}{2}n^a}\} \\ &\geq c\frac{\ln|w| - \ln\frac{3}{2}n^a}{\ln n - \ln\frac{3}{2}n^a} \\ &\geq \frac{c}{\ln n}. \end{aligned}$$

(The second inequality uses a standard result about Brownian motion; see, for example, [7, Section 1.7].) Note that given event X, the Wiener sausage generated by $B[0, T_n]$ can only disconnect 0 from ∂D_n at some point outside the ball of radius n^a.

Next we want to find lower bounds on the probability that, given event X, the Wiener sausage starting at a point $x \in \partial D_{n^a}$ does not disconnect 0. We will make use of the following lemma.

LEMMA 3.2: *For every $a > 0$, there exist $\epsilon > 0$ and $c > 0$ such that for $y \in \partial D_{n^a + \kappa \ln n}$ and for $x \in \partial D_{n^a}$,*

$$P^y(\bar{W}_n) \leq P^x(\bar{W}_n) + cn^{-(1-a)\gamma - \epsilon}.$$

Proof: Let V be the event

$$\{|B(t)| > n^a,\ t \in [0, T_n]\}.$$

Then

$$\begin{aligned} P^y(\bar{W}_n) &= P^y(\bar{W}_n \cap V) + P^y(\bar{W}_n \cap V^c) \\ &\leq P^y(\bar{W}_n \cap V) + P^x(\bar{W}_n). \end{aligned}$$

Let $b > a$, where the exact value of b will be determined later. Note that by the strong Markov property, for $z \in \partial D_{n^b}$,

$$P^y(\bar{W}_n \cap V) \leq P^y\{T_{n^b} < T_{n^a}\}\ P^z(\bar{W}_n \cap V) \leq P^y\{T_{n^b} < T_{n^a}\}\ P^z(\bar{W}_n).$$

We note that

$$\begin{aligned} P^y\{T_{n^b} < T_{n^a}\} &= \frac{\ln|y| - \ln n^a}{\ln n^b - \ln n^a} \\ &\leq \frac{(n^{-a}\kappa \ln n + \ln n^a) - \ln n^a}{(b-a)\ln n} \\ &= \frac{\kappa}{b-a}\, n^{-a}. \end{aligned}$$

Also,

$$P^z(\bar{W}_n) \le \phi(n^{1-b}) \le n^{-(1-b)\gamma+\delta},$$

for any $\delta > 0$ given n sufficiently large (depending on δ). Thus, we have

$$P^y(\bar{W}_n) \le P^x(\bar{W}_n) + c(b-a)^{-1}n^{-a}n^{-(1-b)\gamma+\delta}.$$

To conclude the proof, we choose b and δ so that $a + (1-b)\gamma - \delta > (1-a)\gamma$. ∎

From this lemma and the strong Markov property, we can see that if $x \in \partial D_{n^a}$ and $y \in \partial D_{n^a+\kappa \ln n}$,

$$P^x(\bar{W}_n \mid X^c) \le P^y(\bar{W}_n) \le P^x(\bar{W}_n) + c_1 n^{-(1-a)\gamma-\epsilon}.$$

So

$$\begin{aligned} P^x(\bar{W}_n \mid X) &= [P^x(\bar{W}_n) - P^x(\bar{W}_n \mid X^c)P^x(X^c)][P^x(X)]^{-1} \\ &\ge [P^x(\bar{W}_n) - (P^x(\bar{W}_n) + c_1 n^{-(1-a)\gamma-\epsilon})(1 - P^x(X))] \\ &\quad \cdot [P^x(X)]^{-1} \\ &\ge [P^x(X)P^x(\bar{W}_n) - c_1 n^{-(1-a)\gamma-\epsilon}][P^x(X)]^{-1} \\ &\ge P^x(\bar{W}_n) - c_1 n^{-(1-a)\gamma-\epsilon}\ln n \\ &\ge cP^x(\bar{W}_n). \end{aligned}$$

Now we are ready to pull the pieces together to get a lower bound for $P^0(\bar{S}_n)$. Define the event

$$\bar{A}_n = \{\max_{0\le t\le \bar{T}_n} \sup_{|t-s|\le \frac{1}{2}} |S(t) - B(s)| \le \kappa \ln n\}.$$

Then, for $x \in \partial D_{n^a}$, using Lemma 2.2,

$$\begin{aligned} P^0(\bar{S}_n) &\ge P^0(U \cap \bar{A}_n \cap X \cap \bar{W}_n) \\ &\ge P^0(U \cap X \cap \bar{W}_n) - P(\bar{A}_n^c) \\ &\ge P^0(U)\, P^x(X \cap \bar{W}_n) - cn^{-100} \\ &= P^0(U)\, P^x(\bar{W}_n \mid X)\, P^x(X) - cn^{-100} \\ &\ge c(n^{-a}P^x(\bar{W}_n)(\ln n)^{-1} - n^{-100}). \end{aligned}$$

Now, for any $\epsilon > 0$, we can find constants $c_1 = c_1(\epsilon)$ and $c_2 = c_2(\epsilon)$ such that

$$\phi(r) \le c_1 r^{-\gamma+\epsilon} \quad \text{and} \quad \phi(r) \ge c_2 r^{-\gamma-\epsilon}.$$

Then using Lemma 3.1, we can find a constant c such that

$$P^0(\bar{S}_n) \geq c n^{-a} n^{-(1-a)\gamma} n^{-a(\gamma/2)} (\ln n)^{-1}.$$

As this holds for all $0 < a < 1$, we have that

$$\limsup_{n\to\infty} -\frac{\ln P^0(\bar{S}_n)}{\ln n} \leq \gamma.$$

Next we will show that

$$\liminf_{n\to\infty} -\frac{\ln P^0(\bar{S}_n)}{\ln n} \geq \gamma.$$

To prove this, it suffices to show that for any $0 < a < 1$

$$\liminf_{n\to\infty} -\frac{\ln \psi(n^a, n)}{\ln n} \geq \gamma(1-a).$$

As in the definition of the Wiener sausage $W_n[s,t]$, we define a fattened random walk of radius $\kappa \ln n$ about $S[s,t]$:

$$V_n[s,t] = \{y \in R^2 \colon |y - S(m)| \leq \kappa \ln n \text{ for some } m \in [s,t]\}.$$

We then let $\bar{V}_n$ denote the event

$$\{V_n[0,\tau_n] \text{ does not disconnect } 0 \text{ and } \partial C_n\},$$

and for $r < n$, let

$$\bar{\psi}(r,n) = \sup_{x\in\partial C_r} P^x(\bar{V}_n).$$

LEMMA 3.3: *For every $0 < a < 1$, there exists a constant $c > 0$ such that*

$$\bar{\psi}(n^a, n) \leq \phi(n^a, n) + c\, n^{-100}.$$

Proof: Fix $0 < a < 1$. Assume that the random walk S starts at $x \in \partial C_{n^a}$. Using the event $\bar{A}_n$ defined above and Lemma 2.2,

$$\begin{aligned}\bar{\psi}(n^a, n) &\leq \sup_{x\in\partial C_{n^a}} P^x(\bar{V}_n \cap \bar{A}_n) + \sup_{x\in\partial C_{n^a}} P^x(\bar{V}_n \cap \bar{A}_n^c) \\ &\leq \sup_{x\in\partial C_{n^a}} P^x(\bar{V}_n \cap \bar{A}_n) + c\, n^{-100}\end{aligned}$$

for some constant $c > 0$.

Now suppose that $Y_n[0, T_n]$ is the closed Wiener sausage of radius $2\kappa \ln n$ about $B[0, T_n]$. And let $\bar{Y}_n$ be the event that $Y_n[0, T_n]$ does not disconnect 0 and ∂D_n. Then

$$\sup_{x \in \partial C_{n^a}} P^x(\bar{V}_n \cap \bar{A}_n) \le P^x(\bar{Y}_n) \le P^x(\bar{B}_n) = \phi(n^a, n)$$

which concludes the proof. ∎

Lemma 3.4: *For every $0 < a < 1$ and $\epsilon > 0$, there exist constants $c, c', \epsilon' > 0$ such that*

$$\psi((2n)^a, 2n) \le c\, n^{-(1-a)\gamma}\, n^{\epsilon} + c'\, n^{-\epsilon'}\, \psi(n^a, n).$$

Proof: Fix $0 < a < 1$. In the following, each supremum is taken over all points y such that $y \in \partial C_{(2n)^a}$. We have

$$\begin{aligned}\psi((2n)^a, 2n) &= \sup P^y(\bar{S}_{2n}) \\ &\le \sup P^y(\bar{S}_{2n} \cap \bar{V}_n) + \sup P^y(\bar{S}_{2n} \cap \bar{V}_n^c) \\ &\le \bar{\psi}((2n)^a, n) + \sup P^y(\bar{S}_{2n} \cap \bar{V}_n^c).\end{aligned}$$

Let $\bar{Y}_n$ be as above. Then, using Lemma 3.3,

$$\begin{aligned}\bar{\psi}((2n)^a, n) &\le \sup P^y(\bar{V}_n \cap \bar{A}_n) + \sup P^y(\bar{V}_n \cap \bar{A}_n^c) \\ &\le P^y(\bar{Y}_n) + P^y(\bar{A}_n^c) \\ &\le \phi((2n)^a, n) + c\, n^{-100}.\end{aligned}$$

For any $\delta > 0$, we can find a constant $c > 0$ such that $\phi(r) \le c r^{-\gamma+\delta}$, and therefore, for any $\epsilon > 0$,

$$\phi((2n)^a, n) + c\, n^{-100} \le c\, n^{-(1-a)\gamma}\, n^{\epsilon}$$

for some $c > 0$.

Next, we can use an argument similar to that used in the proof of Lemma 3.1 to bound $\sup P^y(\bar{S}_{2n} \cap \bar{V}_n^c)$. Then, by Lemma 2.3, there exist constants $c', \epsilon' > 0$ such that

$$\begin{aligned}\sup P^y(\bar{S}_{2n} \cap \bar{V}_n^c) &\le \psi(2\kappa \ln n, n) \\ &\le \psi(2\kappa \ln n, n^a)\, \psi(n^a, n) \\ &\le c'\, n^{-\epsilon'}\, \psi(n^a, n)\end{aligned}$$

which completes the proof. ∎

If f is a positive, bounded function satisfying

$$f(2n) \leq c_1 n^{-b} + c_2 n^{-\epsilon} f(n)$$

for some positive c_1, c_2, b, ϵ, then one can check that there is a c_3 such that

$$f(2n) \leq c_3 n^{-b}.$$

Using Lemma 3.4, we have

$$\psi(n^a, n) \leq c\, n^{\epsilon}\, n^{-(1-a)\gamma}.$$

Thus we have that

$$\liminf_{n\to\infty} -\frac{\ln \psi(n^a, n)}{\ln n} \geq \gamma(1-a).$$

References

[1] P. Auer, *Some hitting probabilities of random walks on Z^2*, in *Limit Theorems in Probability and Statistics* (L. Berkes, E. Csáki and P. Révész, eds.), North-Holland, Amsterdam, 1990, pp. 9–25.

[2] K. Burdzy and G. Lawler, *Nonintersection exponents for Brownian paths. Part I. Existence and an invariance principle*, Probability Theory and Related Fields **84** (1990), 393–410.

[3] K. Burdzy and G. Lawler, *Nonintersection exponents for Brownian paths. Part II. Estimates and applications to a random fractal*, The Annals of Probability **18** (1990), 981–1009.

[4] K. Burdzy, G. Lawler and T. Polaski, *On the critical exponent for random walk intersections*, Journal of Statistical Physics **56** (1989), 1–12.

[5] M. C. Cranston and T. S. Mountford, *An Extension of a Result of Brudzy and Lawler*, Probability Theory and Related Fields **89** (1991), 487–502.

[6] B. Duplantier, G. Lawler, J.-F. Le Gall and T. Lyons, *The geometry of the Brownian curve*, Bulletin de Sciences Mathématiques 2ᵉ série **117** (1993), 91–106.

[7] R. Durrett, *Brownian Motion and Martingales in Analysis*, Wadsworth, Belmont, CA, 1984.

[8] J. Komlós, P. Major and G. Tusnády, *An approximation theorem of partial sums of independent R.V.'s and the sample DF. II.*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete **34** (1976), 33–58.

[9] G.F. Lawler, *Intersections of Random Walks*, Birkhäuser, Boston, 1991.

[10] E. E. Puckette, *Critical Exponents for Intersections of Random Walks in Dimensions Between 1 and 2*, Ph.D. Dissertation, Department of Mathematics, Duke University, Durham, NC, 1994.

[11] W. Werner, *On Brownian disconnection exponents,* Bernoulli **1** (1995), 371–380.

[12] W. Werner, *Bounds for disconnection exponents,* Electronic Comm. Prob. **1** (1996), paper no. 4.